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Changing gears a bit... Lights, Camera, $S = \int L$!

There are numerous ways to approach dynamics, but one of the most universally powerful (in that it can be used for non-relativistic or relativistic, classical or quantum, particles or fields) is the action principle based on a Lagrangian.

Reviewing the classical particle case we start with (in 2D) $\underbrace{L(x, y, \dot{x}, \dot{y})}_{\text{Lagrangian}}$ and form $\underbrace{S = \int_{t_1}^{t_2} L\,dt}_{\text{action}}$

The action is a functional which essentially means a function of functions.

For example, whereas a function takes an argument and returns a number: $f(x) = x^2 \Rightarrow f(2) = 4$
a functional does the same when we insert a function: $S[f(x)] = \int_0^1 f(x)dx$

$$\Downarrow$$

$$S[x^2] = \int_0^1 x^2 dx = \frac{1}{3}$$

What makes this nice is that in the same way we can extremize a function w/ $\frac{df}{dx} = 0$ to find the value of x which achieves the max or min of $f(x)$, we can also extremize a functional w/ $\frac{\delta S}{\delta f} = 0$ to find the function $f(x)$ which achieves the max or min of $S[f(x)]$.

However what is most relevant for physics is that extremizing the action functional $S = \int L$ leads to conditions (equations of motion) which correctly describe the behavior of physical systems. This is not obvious and actually had to wait until quantum mechanics to be understood.

So let's do this for our 2D point particle.

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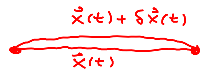
$$S[x(t), y(t)] = \int_{t_1}^{t_2} L(x, y, \dot{x}, \dot{y})\, dt$$

or

$$S[\vec{x}(t)] = \int_{t_1}^{t_2} L(\vec{x}, \dot{\vec{x}})\, dt$$

What we want to do is consider how $S[\,]$ changes to first order as we vary the function $\vec{x}$.
Consider $\vec{x}(t) = \vec{x}(t) + \delta\vec{x}(t)$ where $\delta\vec{x}(t_1) = \delta\vec{x}(t_2) = 0$.



Then to first order: $$\delta S = \int_{t_1}^{t_2} \left(\delta\vec{x}\cdot\frac{\partial L}{\partial \vec{x}} + \underbrace{\delta\dot{\vec{x}}\cdot\frac{\partial L}{\partial \dot{\vec{x}}}}\right) dt = 0$$

$$\int_{t_1}^{t_2} \left(L(\vec{x}+\delta\vec{x}, \dot{\vec{x}} + \delta\dot{\vec{x}}) - L(\vec{x}, \dot{\vec{x}}) \right) dt$$

Take this term and use: $\frac{d}{dt}\left(\delta\vec{x}\cdot\frac{\partial L}{\partial \dot{\vec{x}}}\right) = \delta\dot{\vec{x}}\cdot\frac{\partial L}{\partial \dot{\vec{x}}} + \delta\vec{x}\cdot\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right)$

to integrate by parts: $$\int_{t_1}^{t_2} \delta\dot{\vec{x}}\cdot\frac{\partial L}{\partial \dot{\vec{x}}}\, dt = \underbrace{\left(\delta\vec{x}\cdot\frac{\partial L}{\partial \dot{\vec{x}}}\right)\Big|_{t_1}^{t_2}}_{=0 \text{ since } \delta\vec{x}(t_1) = \delta\vec{x}(t_2) = 0} - \int_{t_1}^{t_2} \delta\vec{x}\cdot\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) dt$$

Then we have: $$\delta S = \int_{t_1}^{t_2} \underbrace{\delta\vec{x}}_{\text{this is arbitrary so}} \cdot \underbrace{\left(\frac{\partial L}{\partial \vec{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right)\right)}_{=0} dt = 0$$

Leading to the Euler-Lagrange equations of motion: $$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$
$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$$

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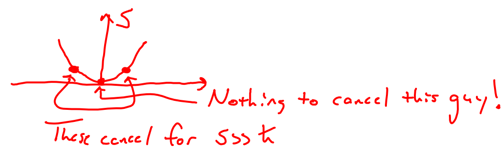
In classical physics we solve the EL equations w/ boundary conditions to determine the behavior of the system.

In quantum physics we start w/ $S=\int_{t_1}^{t_2} L\,dt$ and evaluate $e^{i\frac{S}{\hbar}}$ for <u>all</u> paths $\vec{x}(t)+\delta\vec{x}(t)$ w/ $\delta\vec{x}(t_1)=\delta\vec{x}(t_2)=0$

Then we add together the results $\sum_k e^{i\frac{S_k}{\hbar}}$ and then $\left|\sum_k e^{i\frac{S_k}{\hbar}}\right|$ is the probability for a particle to actually move between the prescribed starting and ending points.

Since this is a (usually continuous) sum over all paths, we call it a <u>path integral</u>.

The classical limit is when $\hbar\to 0$ or more appropriately $S\gg\hbar$ in which case destructive interference between paths other than those which are <u>stationary</u>, i.e. satisfy $\delta S=0$. But that's classical!



To relativize and field theorize things we consider: $S[\phi(x^\mu), \underbrace{\frac{\partial \phi}{\partial x^\mu}}_{\partial_\mu \phi}] = \int_{\phi(x^\mu_i)}^{\phi(x^\mu_f)} \mathcal{L}(\phi, \partial_\mu \phi)\, \underbrace{d^4x}$

No reason to treat time as special!

We won't go through the derivation, but it looks very similar.

In the end you could almost guess the form of the EL equations:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$
$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) \Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0}$$

To make sense of this first observe that $\frac{d}{dt} \to \frac{\partial}{\partial x^\mu}$ as might be expected (relativizing things). Also recall that for point particles the degrees of freedom are $\vec{x}(t)$, hence $\frac{\partial L}{\partial x}$, etc. whereas for fields the degrees of freedom are $\phi(x^\mu)$, hence $\frac{\partial \mathcal{L}}{\partial \phi}$ (particles $\to$ fields).

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If you are familiar with Lagrangian mechanics then you are probably used to constructing the Lagrangian with the kinetic and potential energy of the degrees of freedom according to

$$L = \underbrace{T}_{\text{kinetic energy}} - \underbrace{V}_{\text{Potential energy}}$$

The idea of potential energy is useful, but for our purposes it is better to directly associate potential energies with the interactions between things (fields in our case).

So generically we expect the Lagrangian to split into: $\mathcal{L} = \mathcal{L}_{kinetic} + \mathcal{L}_{interaction}$

This sign is not that important since we haven't yet specified how to construct $\mathcal{L}_{interaction}$

Now if we have no interactions, then we have what is called a free theory in which case $\mathcal{L}_{free} = \mathcal{L}_{kinetic}$ so sometimes we call the kinetic term the "free Lagrangian".

The program we will follow is to first consider free Lagrangians and then introduce interactions through the principle of local gauge invariance.

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Free Lagrangians

In classical point particle physics you would say $L_{free} = T = \frac{p^2}{2m} = \frac{1}{2}mv^2$.

However this is based on a massive scalar particle and built from the 3-momentum $\vec{p}$ (or velocity $\vec{v}$).

All of this changes for (possibly massless) relativistic fields.

Fortunately we only have 3 cases to consider: $\underbrace{\text{spin-0}}_{\text{Higgs}}$, $\underbrace{\text{spin-}\frac{1}{2}}_{\text{matter}}$, $\underbrace{\text{spin-1}}_{\text{force particles}}$

We won't derive these free Lagrangians. Doing so can be done in various ways at different levels of sophistication and to be honest several of them were actually guessed in their original discovery. We will just present them one at a time, then derive the Euler-Lagrange equation of motion.

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Spin-0 (scalars) ϕ

$$\mathcal{L}_{Free} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2$$

Notice that the mass term is separate which allows us to consistently handle $m=0$ cases.

$$= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

$$\left(\frac{mc}{\hbar}\right)^2\phi - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial}{\partial x^\nu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right) = 0$$

$$\left(\frac{mc}{\hbar}\right)^2\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{1}{2}\eta^{\mu\nu}\partial_\nu\partial_\mu\phi = 0$$

$$\left(\frac{mc}{\hbar}\right)^2\phi - \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

$$\partial_\mu\partial^\mu\phi - \left(\frac{mc}{\hbar}\right)^2\phi = 0$$ The Klein-Gordon Equation

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Spin-1 (vectors) A^μ

$$\mathcal{L}_{free} = \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\mu A_\mu \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \text{ and } F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\rho}F_{\lambda\rho}$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{1}{4\pi}\left(\frac{mc}{\hbar}\right)^2 A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{1}{4\pi}F^{\mu\nu}$$ (You get to fill in the details in your homework)

Then: $\partial_\mu F^{\mu\nu} - \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0$ The Proca Equation

Taking $m^2 = 0$ we have $\partial_\mu F^{\mu\nu} = 0 \Rightarrow \begin{cases} \vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = 0 & \text{usually } \vec{J} \\ \vec{\nabla}\cdot\vec{E} = 0 & \text{usually } \rho \end{cases}$ ½ of Maxwell's equations

Note: $F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$

$\left.\begin{matrix} \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \end{matrix}\right\}$ Non-dynamical, come from electromagnetic "geometry"

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Spin-$\frac{1}{2}$ (spinors) ψ & $\bar{\psi}$

$\mathcal{L}_{free} = (\hbar c)\bar{\psi}\gamma^\mu\partial_\mu\psi + mc^2\bar{\psi}\psi$ We treat ψ & $\bar{\psi}$ as independent degrees of freedom for reasons to be discussed.

Varying w.r.t. $\bar{\psi}$:

$$\underbrace{\frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right)}_{\hbar \underbrace{\gamma^\mu\partial_\mu}_{\equiv \not{\partial}}\psi + mc^2\psi} = 0$$

$$\hbar \underbrace{\gamma^\mu\partial_\mu}_{\equiv\not{\partial}}\psi + mc^2\psi = 0 \Rightarrow \not{\partial}\psi + \frac{mc}{\hbar}\psi = 0$$ The Dirac Equation

Interesting...

Varying w.r.t. ψ:

$$\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = 0$$

$$mc^2\bar{\psi} - \partial_\mu\left(\hbar c\bar{\psi}\gamma^\mu\right) = 0 = \not{\partial}\bar{\psi} - \frac{mc}{\hbar}\bar{\psi} = 0$$ (The adjoint of the Dirac Equation)